References

[1]Fox, R. L., and Kapoor, M. D., "Rates of Change of Eigenvalues and Eigenvectors," *AIAA Journal*, Vol. 6, No. 12, 1968, pp. 2426–2429.

[2]Rogers, L. C., "Derivatives of Eigenvalues and Eigenvectors," *AIAA Journal*, Vol. 8, No. 5, 1970, pp. 943, 944.

[3]Nelson, R. B., "Simplified Calculation of Eigenvector Derivatives," *AIAA Journal*, Vol. 14, 1976, pp. 1201–1205.

[4]Ojalvo, I. U., "Efficient Computation of Modal Sensitivities for Systems with Repeated Frequencies," *AIAA Journal*, Vol. 26, No. 3, 1988, pp. 361–366.

[5]Dailey, R. L., "Eigenvector Derivatives with Repeated Eigenvalues," *AIAA Journal*, Vol. 27, No. 4, 1989, pp. 486–491.

[6]Mills-Curran, W. C., "Comment on Eigenvector Derivatives with Repeated Eigenvalues," *AIAA Journal*, Vol. 28, No. 10, 1990, p. 1846.

[7]Mills-Curran, W. C., "Calculation of Eigenvector Derivatives for Structures with Repeated Eigenvalues," *AIAA Journal*, Vol. 26, No. 7, 1988, pp. 867–871.

[8]Chen, T. Y., "Design Sensitivity Analysis for Repeated Eigenvalues in Structural Design," *AIAA Journal*, Vol. 31, No. 12, 1993, pp. 2347–2350.

Topology Optimization with Superelements

R. J. Yang* and C. M. Lu†

Ford Motor Company, Dearborn, Michigan 48121

Introduction

PREVIOUS research on topology optimization focused on material microstructure modeling and efficient optimization techniques.[1–5] In general, the entire structure is analyzed and subsequently optimized. However, in a real design environment, it often occurs that only a small part of a large structure is allowed to change repeatedly and the rest is kept unchanged. Significant savings can be achieved if the analysis code takes this into account. The superelement or substructure formulation used in the finite element method is most suitable for this purpose.

Optimal design of large structures with substructuring or superelements was discussed in the literature.[6–8] Botkin and Yang[9] identified that tremendous savings can be achieved in both finite element analysis and design sensitivity analysis for three-dimensional shape optimization. In topology optimization, the computational advantages are similar to those in shape optimization. Tenek and Hagiwara[10] used a substructure method for obtaining optimum vehicle body panel topologies. Two iteration loops were proposed. The outer loop was the analysis loop that was concerned with the displacement field of the full vehicle finite element analysis. The inner loop was the topology optimization loop for the target panel that was removed from the vehicle structure. Since the target panel was totally disconnected from the full body during the topology optimization process, the convergence was not as good as that for considering full structure.

In this research, the superelement method was employed for finite element analysis. The target panel was modeled as the residual structure and the rest as one big substructure. Unlike the approach of Tenek and Hagiwara,[10] only one optimization loop is required and no analysis approximation is made during optimization iterations. This results in a better convergence characteristic and a more efficient optimization process as opposed to that for the full structure analysis.

Received June 10, 1995; revision received Nov. 20, 1995; accepted for publication Nov. 27, 1995.

*Staff Technical Specialist, Computer-Aided Engineering Department, Ford Research Laboratories, P.O. Box 2053, MD 2122-SRL. Member AIAA.

†Research Engineer, Computer-Aided Engineering Department, Ford Research Laboratories, P.O. Box 2053, MD 2122-SRL. Member AIAA.

Superelement Method

The superelement method first divides the structure into a number of smaller substructures. For each substructure the nodes that are common to adjoining substructures are called boundary nodes. The degrees of freedom of these nodes are called boundary degrees of freedom. The nodes that are not at the boundary of a substructure are called interior nodes. The associated degrees of freedom are called interior degrees of freedom. Conceptually, a reduced set of equilibrium equations is derived in terms of displacements of the boundary nodes for the entire structure. This is accomplished by eliminating interior degrees of freedom for all substructures from the governing equation. The reduced set of equations is solved for boundary displacements. The interior displacements are then computed from boundary displacements.

Consider one substructure; the equilibrium equation is written as

$$\begin{bmatrix} K_{BB} & K_{IB} \\ K_{BI} & K_{II} \end{bmatrix} \begin{bmatrix} z_B \\ z_I \end{bmatrix} = \begin{bmatrix} F_B \\ F_I \end{bmatrix} \tag{1}$$

where K is the stiffness matrix, F is the load vector, and B and I indicate the boundary and interior quantities, respectively. The degrees of freedom for the interior and boundary nodes can be separated as

$$z_I = -K_{II}^{-1} K_{IB} z_B + K_{II}^{-1} F_I \tag{2}$$

$$z_B = -K_{BB}^{-1} K_{BI} z_I + K_{BB}^{-1} F_B \tag{3}$$

Substituting Eq. (2) into Eq. (3), the interior degrees of freedom are eliminated as

$$\left[K_{BB} - K_{BI} K_{II}^{-1} K_{IB}\right] z_B = F_B - K_{BI} K_{II}^{-1} F_I \tag{4}$$

Note that in Eq. (4) the degrees of freedom z_I are condensed to the boundary degrees of freedom z_B for each substructure that is considered as a superelement. Assembling all superelements, a reduced set of the equilibrium equations is obtained and subsequently solved. Although the size of the reduced set is much smaller than that of the entire structure, it is debatable that the superelement formulation has any advantage over the full structure approach for a one-shot analysis. The reason is that, unlike the full structure analysis, additional matrix operations and a complicated database for bookkeeping are required for the superelement analysis. However, in an iterative design environment as in structural optimization, the overhead is easily offset by the reduction of the problem size. Thus, a significant computer savings can be achieved by this formulation in which only the affected or changed substructures are updated.

For topology optimization problems, two superelements are required. The structure that is kept fixed is defined as the substructure α and the structure that is subjected to change as the residual structure r. Before optimization, the stiffness matrix of the substructure α is formulated and the interior degrees of freedom are transformed to the boundary degrees of freedom that are adjoining the residual structure. The reduced set of the equilibrium equations for the residual structure is written as

$$K_{rr} z_r = F_r \tag{5}$$

During the optimization process, only the residual structure r is solved.

Design Sensitivity Analysis

In addition to the computational advantage over the full structure finite element analysis, the superelement method is more efficient in sensitivity calculations. Consider a full structure; the equilibrium equation is

$$K z = F \tag{6}$$

Differentiate Eq. (6) with respect to the design variable b, and the sensitivity of the displacement is written as

$$K \frac{\partial z}{\partial b} = -\frac{\partial K}{\partial b} z + \frac{\mathrm{d}F}{\mathrm{d}b} \tag{7}$$

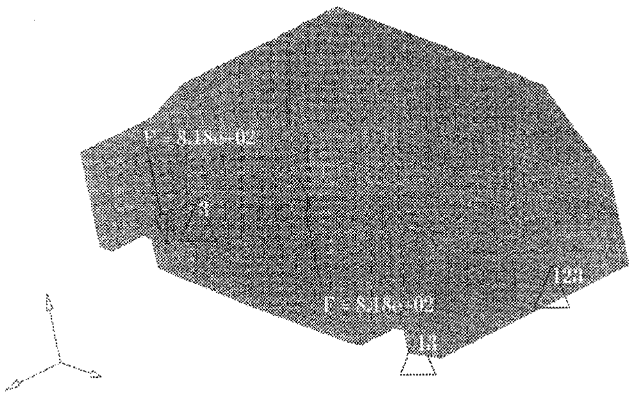

Fig. 1 Simplified body structure.

Note that for each design variable b the system of equations in Eq. (7) needs to be solved once. Although the stiffness matrix K is identical to that in Eq. (6) so that the decomposed matrix can be reused, it is still costly. Instead, if the residual structure is used for the sensitivity analysis, a major computational advantage can be achieved, as the size of the stiffness matrix K_{rr} in Eq. (5) is much smaller than that of K in Eq. (7).

The derivation of Eq. (7) is called the direct method in the literature. Another very popular approach for the sensitivity analysis is the adjoint variable method employed in this study.[5,11] Similar computational advantage can be achieved, as it requires to solve a system of equations with the identical stiffness matrix.

Design Example

A simplified body structure with torsional loads and boundary conditions, shown in Fig. 1, is used as an example. The finite element

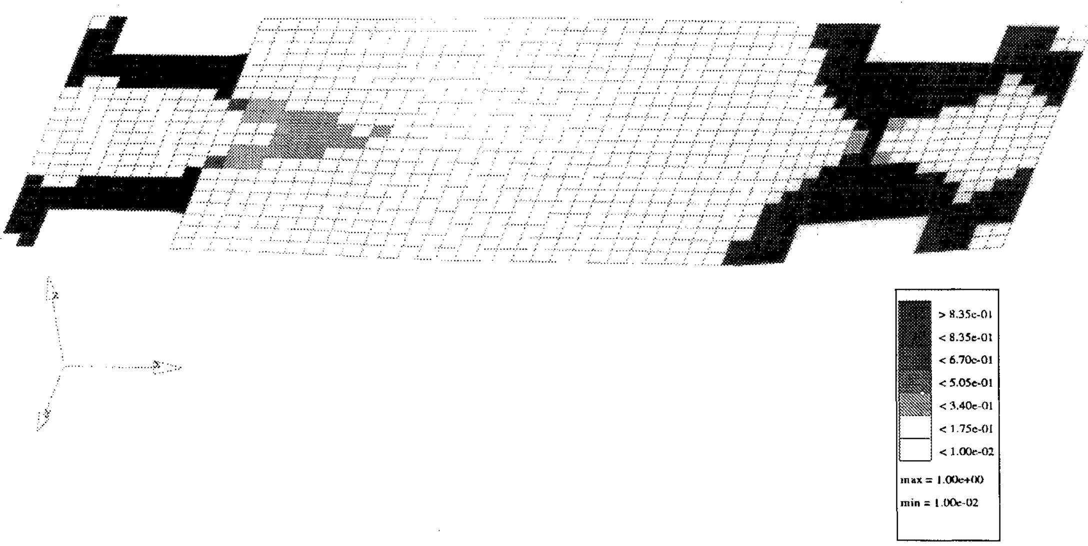

Fig. 2 Topology optimization plot; normalized material density distribution.

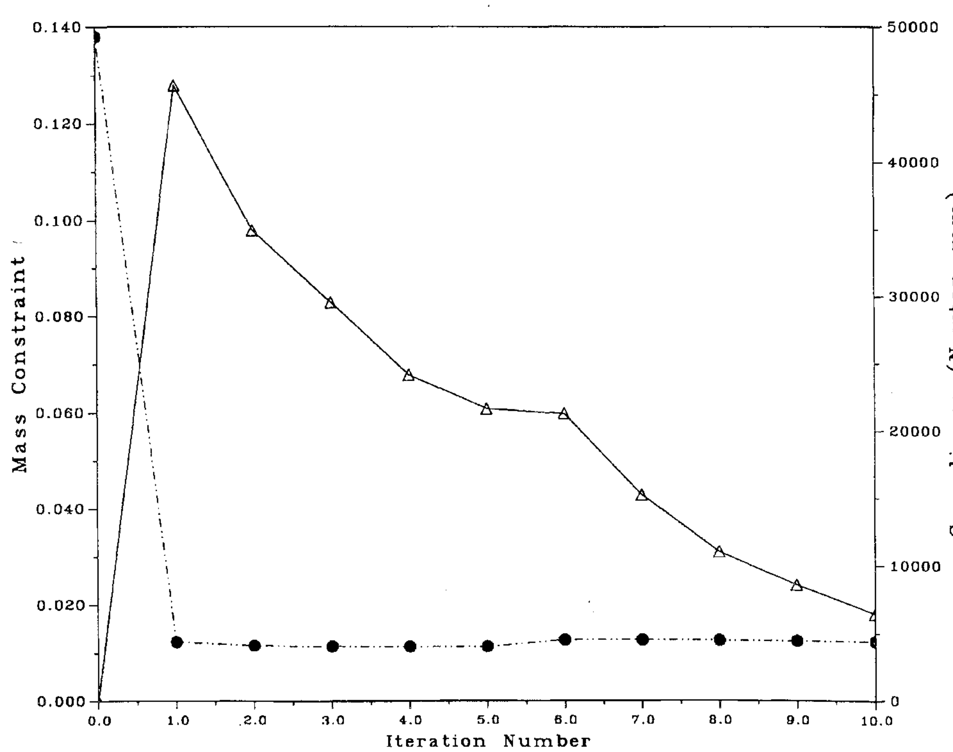

Fig. 3 Design histories: △, mass constraint and ●, compliance.

model contains 5456 QUAD4 and 20 TRIA3 elements, 5508 grid points, and more than 33,000 degrees of freedom. MSC/NASTRAN was used for the finite element analysis.

Assume that the floor pan is the design domain, i.e., only the topology of the floor structure is allowed to change. The floor pan is then selected as the residual structure and the rest as another substructure. The number of design variables that are the element densities in the floor structure is 1024.

The CPU time is first compared for the finite element analysis on which more than 90% of the CPU time is spent for one optimization cycle. The design goal in this case is to minimize the compliance, i.e., maximize the stiffness with a 25% material usage constraint. Note that no additional finite element analyses are required for computing the compliance sensitivities.

The average CPU time for the full structure approach is 80 s on a Cray Y-MP2E. As for the superelement method, the first analysis requires 120 s. The overhead in the first iteration is 40 CPU s for additional matrix operations and database setup. However, only 50 s is required for the subsequent analysis. The factor of savings is more than 1.6 for this example. The normalized material density distribution is shown in Fig. 2. The design histories are shown in Fig. 3. Identical results are obtained for the full structures and the substructure analyses.

The problem was resolved to minimize the displacements at load points; 25% material usage was imposed as the constraint. Using the adjoint variable method, two additional analyses are required for obtaining displacement sensitivities. The average CPU time for the full structure analysis is 92 s, which includes three load cases: one for the real load and two for the adjoint loads. The superelement approach requires 122 s for the first analysis and 54 s for the subsequent analysis. The factor of savings is more than 1.7. Note that the overhead in the first iteration is 30 CPU s, which is less than that in the previous case.

Summary

The superelement formulation used in the finite element method was employed for three-dimensional topology optimization problems. This approach only reformulates the stiffness matrix for the modified part so that it can provide tremendous savings in computing resources for large optimization problems. A body structure is used as an example to demonstrate the use and efficiency of the formulation.

Acknowledgments

The authors wish to express appreciation to M. Chirehdast for providing the body structure model and M. McNamee of MacNeal–Schwendler Corporation for his technical support on superelement method.

References

[1]Bendsoe, M. P., and Kikuchi, N., "Generating Optimal Topologies in Structural Design Using a Homogenization Method," *Computer Methods in Applied Mechanics and Engineering*, Vol. 71, No. 2, 1988, pp. 197–224.

[2]Diaz, A., and Kikuchi, N., "Solutions to Shape and Topology Eigenvalue Optimization Problems Using a Homogenization Method," *International Journal for Numerical Methods in Engineering*, Vol. 35, No. 7, 1992, pp. 1487–1502.

[3]Rozvany, G. I. N., Zhou, M., and Birker, T., "Generalized Shape Optimization Without Homogenization," *Structural Optimization Journal*, Vol. 4, Nos. 3, 4, 1992, pp. 250–252.

[4]Mlejnek, H. P., and Schirrmacher, R., "An Engineer's Approach to Optimal Material Distribution and Shape Finding," *Computer Methods in Applied Mechanics and Engineering*, Vol. 106, Nos. 1, 2, 1993, pp. 1–26.

[5]Yang, R. J., and Chuang, C. H., "Optimal Topology Design Using Linear Programming," *Computers and Structures*, Vol. 52, No. 2, 1994, pp. 265–275.

[6]Haug, E. J., and Arora, J. S., *Applied Optimal Design*, Wiley, New York, 1979.

[7]Huang, J.-S., "The Optimal Design of Structural Systems by the Superelement Method," *Proceedings of the MSC 1988 World Users Conference*, Vol. 1, No. 7, MacNeal–Schwendler, 1988, pp. 1–8.

[8]Patel, H. D., "Multidisciplinary Design Optimization with Superelements in MSC/NASTRAN," *Proceedings of the 4th AIAA/USAF/NASA/OAI Symposium on Multidisciplinary Analysis and Optimization*, AIAA, Washington, DC, 1992, pp. 340–348.

[9]Botkin, M. E., and Yang, R. J., "Three-Dimensional Shape Optimization with Substructuring," *AIAA Journal*, Vol. 29, No. 3, 1991, pp. 486–488.

[10]Tenek, L. H., and Hagiwara, I., "A Substructure Method Incorporating Homogenization for Finding Optimum Vehicle Body Panel Topologies," *JSME International Journal*, Vol. 38, No. 1, 1995, pp. 44–51.

[11]Haug, E. J., Choi, K. K., and Komkov, V., *Design Sensitivity Analysis of Structural System*, Academic, New York, 1986.